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On solutions of crack surface opening displacement of a penny-shaped crack in an elastic cylinder subject to tensile loading

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A simple analytical expression for the surface displacement of a penny-shaped crack in an elastic cylinder subject to remote tensile loading is proposed based on a modified shear-lag model. The results are then compared with the dilute solution [1] and those of finite element calculation. It is found that the present work gives much better result than the dilute model.

1. Introduction

The problem of an elastic solid containing a penny-shaped crack subject to remote loading forms an important part of analytical fracture mechanics. Survey of early works can be found in the representative books (e.g., Kassir and Sih [2]). The general treatments are based on the integral transformation theory and the complex variable theory. The results are expressed in the form of integral expressions and only numerical results are presented. On the other hand, microcracking is one of the most important damage mechanisms in engineering materials. To build the constitutive equations of the microcracking material it is essential to obtain the simple analytical expressions of the variation of crack surface opening displacement with the remote loading. Budiansky and O'Connell [1] made an investigation of an isolated penny-shaped crack in an infinite elastic medium subject to remote loading by applying Eshebly's theorem and the self-consistent method. It is therefore desired for practical use that a simple expression for a crack in a finite domain should be provided.

In studying the bridging effect in fiber-reinforced composites, McCartney [3] formulated an approximate solution for fiber-bridging and fiber-cracking problems. In general, his solution may be thought as construction of an admissible stress field and therefore provides an upper bound result. A modified shear-lag model was then proposed based on his work [4, 5]. Here a new material parameter G_{I} called "interfacial shear modulus" was introduced which describes the influence of shear deformation in the matrix above the slipping region. The determination of G_{I} is based on McCartney's approximate solution. A preliminary calculation shows that the new shear-lag model could give a simple expression for axial stresses and interfacial shear stress which coincide well with the two-dimensional analysis [4].

In the present study we first make a precision of the adopted two-dimensional approximate solution. Then we make a detailed finite element calculation to verify the result of modified shear-lag model and generalize that model to the microcracking problem in one elastic medium. A comparison with the finite element calculation shows that the present model provides a very good approximation for microcracking elastic bodies.

2. Modified shear-lag model

Consider a composite cylinder of length L subject to remote uniform applied axial stress σ_0. The fiber of the cylinder cracks at $z = 0$ (Fig. 1). Let σ_{f}, u_{f} and σ_{m}, u_{m} be the axial stress and displacement in the fiber and the matrix, R the radius of the fiber, V_{f} and V_{m} the volume fraction of the fiber and the matrix. The axial equilibrium equation of the fiber and the matrix become

$$\frac{\mathrm{d}\sigma_{\mathrm{f}}}{\mathrm{d}z} + \frac{2}{R}\tau_{\mathrm{s}} = 0 \tag{1a}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}(V_{\mathrm{f}}\sigma_{\mathrm{f}} + V_{\mathrm{m}}\sigma_{\mathrm{m}}) = 0 \tag{1b}$$

The interfacial shear stress τ_{s} follows the bilinear law

$$\tau_{\mathrm{s}} = \begin{cases} \dfrac{G_{\mathrm{I}}}{R}\Delta, & \Delta < \dfrac{R}{G_{\mathrm{I}}}\tau_0 \\ \tau_0, & \Delta \geq \dfrac{R}{G_{\mathrm{I}}}\tau_0 \end{cases} \tag{2}$$

where $\Delta = u_{\mathrm{m}} - u_{\mathrm{f}}$, G_{I} is the interfacial shear modulus which depends on the moduli of the matrix and the fiber. The determination of G_{I} follows the work of McCartney

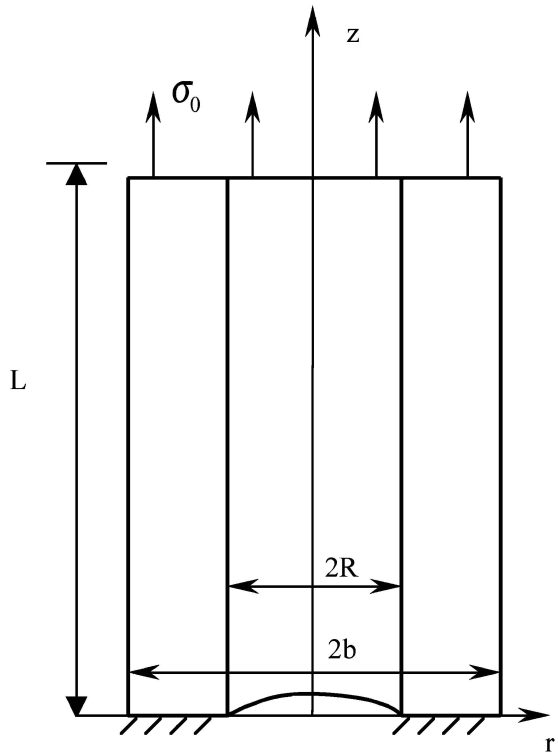

Figure 1 A matrix and broken-fiber composite cylinder model.

[3] and will be outlined later. The axial stress-strain relation of the fiber and the matrix are written as

$$\sigma_{\mathrm{f}} = E_{\mathrm{f}}\frac{\mathrm{d}u_{\mathrm{f}}}{\mathrm{d}z}, \quad \sigma_{\mathrm{m}} = E_{\mathrm{m}}\frac{\mathrm{d}u_{\mathrm{m}}}{\mathrm{d}z} \tag{3}$$

where E_{f} and E_{m} are Young's moduli of the fiber and the matrix. The boundary conditions are

$$(\sigma_{\mathrm{f}})_{z=0} = 0, \quad (\sigma_{\mathrm{m}})_{z=0} = P = \frac{\sigma_0}{V_{\mathrm{m}}} \tag{4a}$$

$$(\sigma_{\mathrm{f}})_{z=L} = \sigma_{\mathrm{f}}^0 = \frac{E_{\mathrm{f}}}{E_{\mathrm{c}}}\sigma_0, \quad (\sigma_{\mathrm{m}})_{z=L} = \sigma_{\mathrm{m}}^0 = \frac{E_{\mathrm{m}}}{E_{\mathrm{c}}}\sigma_0 \tag{4b}$$

where

$$E_{\mathrm{c}} = E_{\mathrm{f}}V_{\mathrm{f}} + E_{\mathrm{m}}V_{\mathrm{m}} \tag{5}$$

In the present case we consider only the non-slipping case. Let $\tau_0 \to \infty$. The solution of the axial stress of the fiber and the matrix become

$$\sigma_{\mathrm{f}} = \sigma_{\mathrm{f}}^0\frac{1-\exp\left(-\frac{\lambda}{R}z\right)}{1-\exp\left(-\frac{\lambda}{R}L\right)} \tag{6a}$$

$$\sigma_{\mathrm{m}} = \frac{1}{V_{\mathrm{m}}}(\sigma_0 - V_{\mathrm{f}}\sigma_{\mathrm{f}}) \tag{6b}$$

where

$$\lambda = \sqrt{\frac{2G_{\mathrm{I}}E_{\mathrm{c}}}{V_{\mathrm{m}}E_{\mathrm{m}}E_{\mathrm{f}}}} \tag{6c}$$

The additional compliance due to cracking can be calculated as

$$\begin{aligned}\Delta &= \frac{2}{E_{\mathrm{m}}}\int_0^L \left(\sigma_{\mathrm{m}} - \sigma_{\mathrm{m}}^0\right)\mathrm{d}z \\ &= \frac{2V_{\mathrm{f}}\sigma_{\mathrm{f}}^0}{V_{\mathrm{m}}E_{\mathrm{m}}}\left(\frac{R}{\lambda} + \frac{\exp\left(-\frac{\lambda}{R}L\right)}{1-\exp\left(-\frac{\lambda}{R}L\right)}\right)\end{aligned} \tag{7}$$

The determination of λ (thus G_{I}) is based on an energy-equivalent criterion. The elastic energy released due to the formation of the crack is given as

$$\Delta E = \frac{\pi R^2}{V_{\mathrm{f}}}(\sigma_0\Delta - \Gamma_{\mathrm{f}}V_{\mathrm{f}}) \tag{8}$$

where Γ_{f} is the surface energy of the fiber. From the two-dimensional approximate solution with $L/R \to \infty$ [3]

$$\Delta = \xi \cdot \frac{2RV_{\mathrm{f}}\sigma_{\mathrm{f}}^0}{E_{\mathrm{m}}V_{\mathrm{m}}}\left(1 - \frac{2\nu_{\mathrm{m}}\beta}{\alpha}\right)K \tag{9}$$

where $\xi = 1$ in the previous work [3], and

$$K = \begin{cases} \dfrac{2p}{p^2-q^2} & a_1 - b_1 < 0 \\ \dfrac{2}{p} & a_1 - b_1 = 0 \\ \dfrac{2p}{p^2+q^2} & a_1 - b_1 > 0 \end{cases} \tag{10}$$

Here ν_{f} and ν_{m} are Poisson's ratios of the fiber and the matrix, $\alpha, \beta, p, q, a_1, b_1$ are functions of E_{f}, E_{m}, ν_{f}, ν_{m}, V_{m} and the temperature only [3].

In the two-dimensional approximate solution [3], the stress-strain relations and the displacement boundary conditions are approximated in an average sense over the cross-section of the fiber and the matrix. That is, the axial displacement fields can be written as independent of r. Thus

$$(u_{\mathrm{m}})_{z=0} = 0, \quad (u_{\mathrm{f}})_{z=0} = u_0 \tag{11}$$

However, from Eshelby's theorem, the penny-shaped crack should be opened to an ellipsoid rather than a disk, that is

$$(u_{\mathrm{f}})_{z=0} = u_0\sqrt{1 - \frac{r^2}{R^2}} \tag{12}$$

Notice that the strain energy change due to the formation of the crack and thus the value of Δ is directly proportional to $(u_f)_{z=0}$. The factor ξ may be modified as

$$\xi = \frac{1}{\pi R^2}\int_{A_{\mathrm{C}}}(1 - r^2/R^2)^{\frac{1}{2}}\mathrm{d}A = \frac{2}{3} \tag{13}$$

where A_{C} is the cross-sectional area of the crack at $z = 0$.

Comparison of (9) with (7) ($L/R \to \infty$) yields

$$\lambda = \frac{1}{\left(1 - \frac{2\nu_m \beta}{\alpha}\right)\xi K} \tag{14}$$

The average crack surface opening displacement can be calculated from Δ

$$\delta = \left(\frac{b}{R}\right)^2 \Delta$$

3. Results

Based on (6) and (14), the axial stress and displacement fields in the matrix are calculated with one-dimensional model. Here

$$\Delta u_m = \frac{1}{E_m} \int_0^z \left(\sigma_m - \sigma_m^0\right) dz \tag{15}$$

The thermal and residual stress effects are neglected in the calculation. The results are then compared with finite element calculations (Figs 2 and 3). Here the solid curves indicate the result of the present model, and the vacuum dots represent the finite element result. It is seen that good agreements are obtained.

With the equivalence of the matrix and fiber properties, the above shear-lag model is extended to the penny-shaped crack analysis in one elastic medium. Figs 4 and 5 show the result for δ. It seems that the present model provides a very good approximation both for infinite

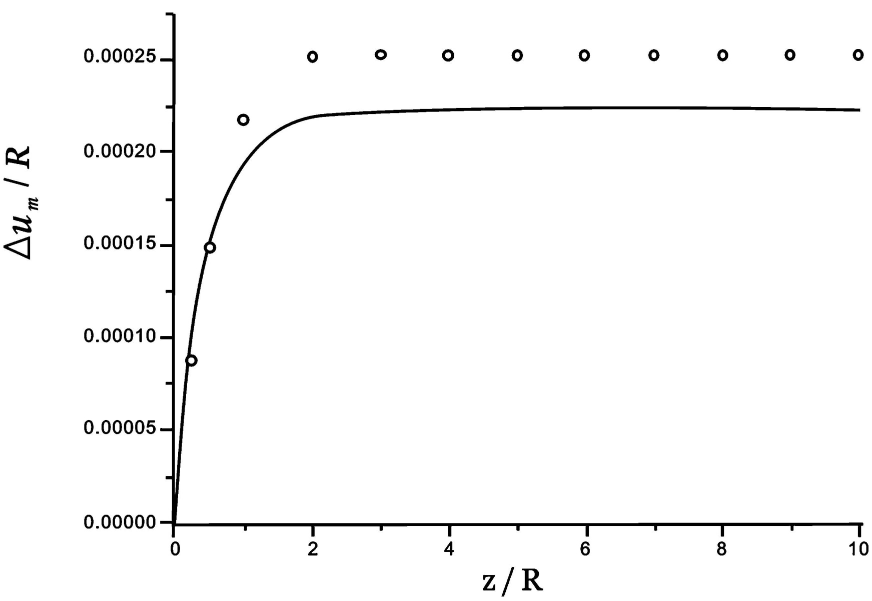

Figure 2 Matrix axial displacement increment Δu_m along z axis. $R/b = 0.707$, $E_f = 98$ GN/m^2, $E_m = 207$ GN/m^2, $\nu_f = 0.25$, $\nu_m = 0.2$, $\sigma_0/E_c = 10^{-3}$, $L/b \to \infty$. (—): Modified shear-lag model and (○): Finite element result.

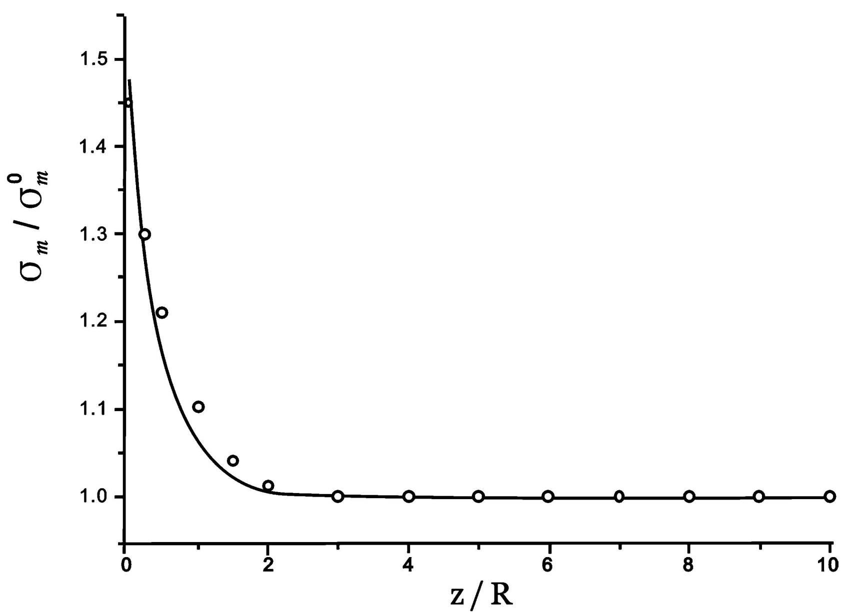

Figure 3 Matrix axial stress σ_m along z axis. $R/b = 0.707$, $E_f = 98$ GN/m^2, $E_m = 207$ GN/m^2, $\nu_f = 0.25$, $\nu_m = 0.2$, $\sigma_0/E_c = 10^{-3}$, $L/b \to \infty$. (—): Modified shear-lag model and (○): Finite element result.

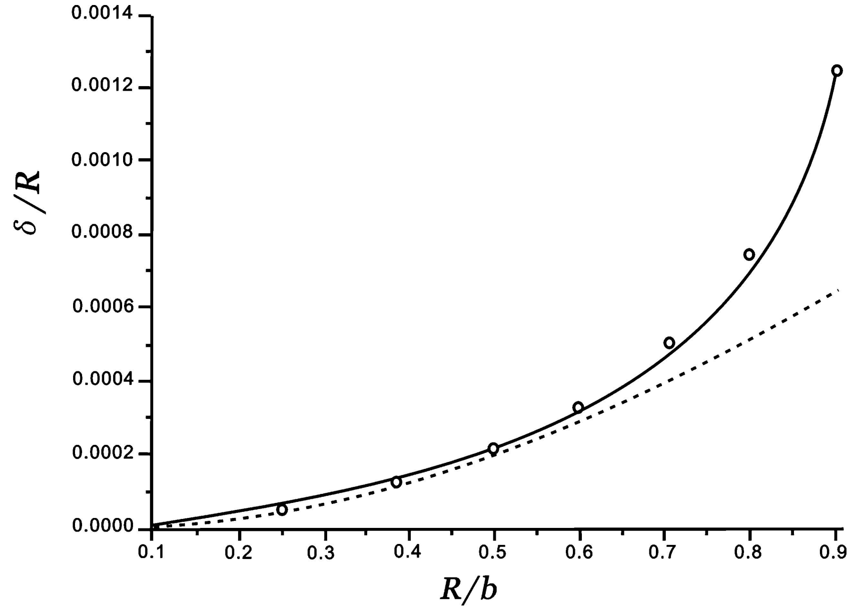

Figure 4 Variation of crack surface opening displacement δ with crack radius R $E_f = E_m = 98$ GN/m^2, $\nu_f = \nu_m = 0.25$, $\sigma_0/E_c = 10^{-3}$, $L/b \to \infty$. (—): Modified shear-lag model, (○): Finite element result, and (- - -): Model of Budiansky and O'Connell [1].

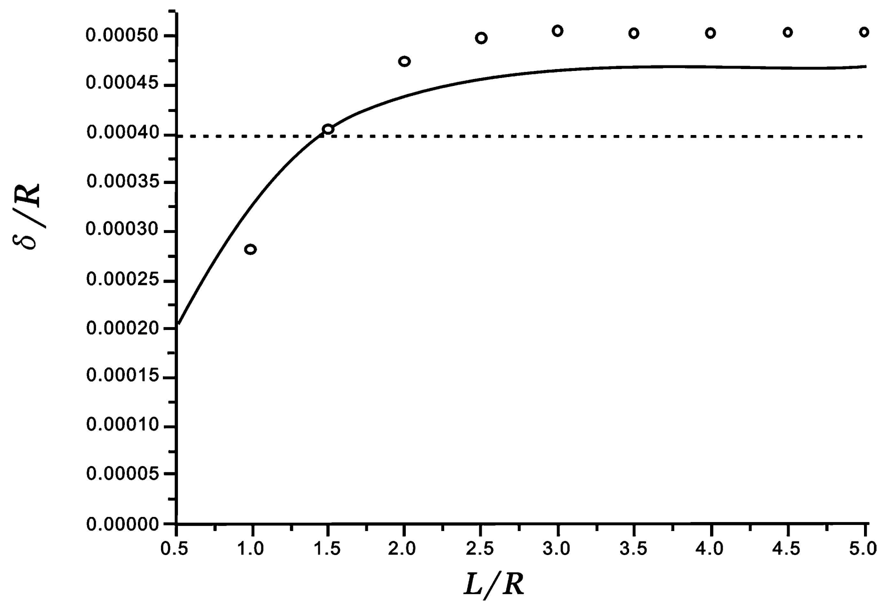

Figure 5 Variation of crack surface opening displacement δ with crack spacing L. $R/b = 0.707$, $E_f = E_m = 98$ GN/m^2, $\nu_f = \nu_m = 0.25$, $\sigma_0/E_c = 10^{-3}$. (—): Modified shear-lag model, (○): Finite element result, and (- - -): Model of Budiansky and O'Connell [1].

and finite domains. It is also found that the model of Budiansky and O'Connell [1] may give quite discrepant results when the cracks get much closer.

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